Semestral Examination, Differential Geometry, 2002

Attempt all questions. Each question carries 15 marks. You can use results proved in class, and needn't furnish proofs for them

1. (i): Consider the 2-form:

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

on $\mathbb{R}^3$. Does there exist a 1-form α such that $\omega = d\alpha$?

(ii): Consider the 2-form $\beta := \omega_{|S^2} = i^*\omega$ where ω is the 2-form in (i) above, and $i : S^2 \subset \mathbb{R}^3$ is the inclusion of the unit sphere centred at the origin. Does there exist a 1-form α on S^2 such that $\beta = d\alpha$?

(iii): Show that the 2-form β of (ii) above is the volume form of S^2 with respect to its Riemannian metric (induced from the Euclidean metric on $\mathbb{R}^3$).

2. Consider the torus obtained by revolving the circle of unit radius centred at $(2, 0, 0)$ about the z-axis. It is defined as the locus:

$$M = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2) = 0\}.$$

(i): Write down a suitable atlas for M, and write down the Riemannian metric induced from the Euclidean metric on $\mathbb{R}^3$ in any one chart of your atlas.

(ii): Compute the second fundamental form of M, again in any one chart of your atlas.

(iii): Write down the scalar (=sectional) curvature function in any one chart of your atlas.

3. (i): Let G be a Lie group, and let $\langle , \rangle$ denote a left- invariant Riemannian metric on G. Let ∇ denote the Levi-Civita connection of this Riemannian metric. Show that for $X, Y, Z \in \mathfrak{g}$, and $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ denoting the corresponding left-invariant vector-fields:

$$\left\langle \nabla_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \right\rangle = \frac{1}{2} \left[\langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle \right].$$

(ii): Show that the path

$$c(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad t \in \mathbb{R}$$

in $G = SL(2, \mathbb{R})$ is not a geodesic with respect to the Levi-Civita connection arising from the left-invariant Riemannian metric defined on G via the inner product $\langle X, Y \rangle = \operatorname{tr} XY^t$ on $\mathfrak{sl}(2, \mathbb{R})$.

(iii): On the other hand, show that the path:

$$c(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad t \in \mathbb{R}$$

is a geodesic with respect to the connection defined in (ii) above.

4. Consider the compact group $G = SU(2)$, with Lie algebra:

$$\mathfrak{su}(2) = \{X \in \mathfrak{gl}(2, \mathbb{C}) : X^* = -X, \ \operatorname{tr} X = 0\}.$$

(i): Show that the symmetric bilinear form $\langle X, Y \rangle := -\operatorname{tr} XY$ on $\mathfrak{su}(2)$ is positive definite, and the left-invariant Riemannian metric on $SU(2)$ defined by this bilinear form is bi-invariant.

(ii): Let ∇ denote the Levi-Civita connection with respect to the bi-invariant Riemannian metric above. Consider the orthonormal basis:

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Compute $\nabla_{X_i} X_j$ for $i, j = 1, 2, 3$ and $i \leq j$.

(iii): Compute the sectional curvatures $K(X_i, X_j)$ for $i, j = 1, 2, 3$ and $i \neq j$.